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1996 J. Phys. A: Math. Gen. 29 4425

(http://iopscience.iop.org/0305-4470/29/15/015)

J. Phys. A: Math. Gen. **29** (1996) 4425–4443. Printed in the UK

Self-similarity and finite-time intermittent effects in turbulent sequences

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Received 4 April 1996

Abstract. We discuss two questions related to the finite-time behaviour of a sequence that generates a self-similar system. The first one concerns the way to approach the self-similarity exponent associated with the system, from the analysis of the sequences observed during a finite time. After this approximation procedure we established a criterion to decide whether the analysed sequence can be considered to be a finite-time subsequence of a generating sequence for a self-similar system. In the second part of this work, assuming the asymptotic self-similarity for the system generated by a sequence, we study the conditions ensuring the appearance of anomalous scaling on the structure functions, due to finite-time effects. We use this method to show that, in the case of a real turbulent sequence, anomalous scaling is not incompatible with asymptotic self-similarity.

Introduction

In this paper we study two questions concerning the self-similarity of an ergodic discrete system, both related to the empirical characterization of self-similarity. First we state what we mean by discrete system and self-similarity. This turns out to be a property of the ergodic measure with respect to a family of observables like the family of the velocity differences in turbulence. In fact the old problem of scaling in fully developed turbulence is the main motivation of this work, although we hope that this approach can also be applied to other natural phenomena where the same kind of effects is encountered, that is why we try to keep some generality in our discussion. Nevertheless, to illustrate all our results we use the physical example of a turbulence sequence, and a mathematical example: the sum of Gaussian random variables, which has well known self-similarity properties. Our approach is sufficiently general to handle these two examples with the same formalism.

We state the problem as follows. Suppose that as time goes to infinity a given sequence generates a self-similar system; then the problem is to detect at finite time this self-similarity. We consider the case where a single self-similarity exponent completely determines the scaling properties of the system. We propose a procedure to compute this exponent from an experimental realization of the system (a single orbit, recorded during a finite time interval) and a criterion to check the validity of the obtained value. This procedure is applied to a moderated Reynolds number turbulent sequence and to a sequence of independent Gaussian random variables with known asymptotic behaviour. In this way we may compare the results and validate the conclusions we obtain.

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0305-4470/96/154425+19$19.50

In the second part we consider the scaling of the structure functions, again in a slightly general framework, although it will be of interest mostly for the turbulence community. We find the relation between the deviations on the maximum value of the observables (the velocity differences for the turbulence sequence), from the exact self-similar behaviour, and the anomalies in the scaling of the structure functions. In this way we show that the structure functions measured from a self-similar system may display anomalous scaling. The deviations on the maximum value, when they are preferentially biased to large values, may be thought of as the superposition of an intermittent perturbation to our original self-similar system, in a way that the perturbation is asymptotically metrically irrelevant. This would imply that the anomalous scaling of structure functions is only perceptible at finite times.

1. Discrete self-similar systems

We consider dynamical systems for which the configuration space is a space of real sequences $\Sigma \subset \mathbb{R}^{\mathbb{N}}$, for instance the space of realizations of a stochastic discrete process or the sequence of fluctuations of a physical real field, experimentally recorded with a specific acquisition frequency. The evolution in time is generated by the action of the time shift on the sequences of the configuration space. The state of the system at time $t \in \mathbb{N}$ is then given by the sequence

$$\boldsymbol{v}^t = \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \ldots \in \Sigma$$

which evolves at time $t+1$ to the state

$$\boldsymbol{v}^{t+1} = \sigma(\boldsymbol{v}^t) = \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \ldots \in \Sigma.$$

We suppose that our system satisfies the following 'ergodic property'†. For any measurable set $\mathcal{B} \subset \Sigma$,

$$\lim_{N\to\infty} \frac{\#\{0 \leqslant t \leqslant N : f \circ \sigma^t(\boldsymbol{v}) \in \mathcal{B}\}}{N+1} = \int_\Sigma f(\boldsymbol{v})\, \mathrm{d}\mu(\boldsymbol{v}) \tag{1.1}$$

independently of the initial condition $\boldsymbol{v}$ in a set $\Sigma^0 \subset \Sigma$ of measure $\mu(\Sigma^0) = 1$. In our case we consider a product sigma-algebra on Σ of copies of the usual Borel sigma-algebra on $\mathbb{R}$. In practice only the sigma-algebra of Borel sets is needed, because the observables we consider take values in $\mathbb{R}$.

The self-similarity of a discrete system is a property of the ergodic measure with respect to a family of observables. In the case of the turbulence sequence we consider the family of velocity differences $\{\Delta_\tau : \Sigma \to \mathbb{R},\ \boldsymbol{v} \mapsto \boldsymbol{v}_\tau - \boldsymbol{v}_1 : \tau \in \mathbb{N}\}$.

In general, the ergodic measure μ is self-similar with respect to a family of observables $\{f_\tau : \Sigma \to \mathbb{R} : \tau \in \mathbb{N}\}$ on the set $T \subset \mathbb{N}$, if for any two values of the parameter $\tau, \tau' \in T$ there exists a rescaling function $\tau/\tau' \mapsto \beta(\tau/\tau')$ such that

$$\mu\{f_{\tau'} \in \mathcal{B}\} = \mu\{f_\tau \in \beta(\tau/\tau')\mathcal{B}\}$$

for any measurable set $\mathcal{B} \subset \Sigma$. This is equivalent to saying that the asymptotic frequency for $f_{\tau'}$ to take values on the set $\mathcal{B}$ is exactly the same as the frequency for f_τ to take a value on the rescaled set $\beta(\tau/\tau')\mathcal{B}$, for $\tau, \tau' \in T$. In this way the observation of the system through the function $f_{\tau'}$ is equivalent, up to a renormalization, to its observation through f_τ. Equation (1.2) defines the 'similarity property'. We call 'similarity range' the range T

† About dynamical systems and ergodicity see, for instance, Halmos [1].

where (1.2) is valid and 'scaling function' the map β which we extend to the whole positive real axis $\mathbb{R}^+$.

In a self-similar system it is sufficient to know the temporal distribution of the f_τ values for a single value of the parameter τ, to determine the temporal distribution of the f_τ values for any other $\tau \in T$, by applying the scale change defined by β. In more generality it is possible to consider scale changes depending on the initial condition (an orbit-dependent rescaling), or rescaling factors depending not on the quotient τ/τ' but on both parameters τ and τ'. In the first case we are dealing with a measure which is not ergodic, for which there are disjoint classes of orbits showing different asymptotic behaviours. The former may be compatible with an ergodic measure and the reason why we restrict our study to the case of a scaling function depending only on the quotient τ/τ', stresses the invariance under changes of time unit on the underlying continuous-time system that, in the case of a sequence of fluctuations of a physical field, we model by a discrete-time one.

Proposition 1. If the temporal distribution of f_τ values is not concentrated at the origin, the scaling function follows a power law behaviour, that is, $\beta(\lambda) = \lambda^\alpha$ for some $\alpha \in \mathbb{R}$.

We prove this proposition in the appendix. From now on we place ourselves in this case, so our scaling function will always be of the form $\beta(\lambda) = \lambda^\alpha$, where α is the 'similarity exponent'.

The classical example of a self-similar discrete system is a sum of independent random variables with the same zero-mean Gaussian distribution. In this case the states of the system are all the realizations of a sequence of independent zero-mean Gaussian random variables, and the family of observables is the family of sums $\{s_\tau : \Sigma \to \mathbb{R} : \tau \in \mathbb{N}\}$ such that $s_\tau(\boldsymbol{v}) = \sum_{t=0}^{\tau-1} \boldsymbol{v}_t$. The self-similarity comes from the stability of the Gaussian measures under convolutions and in this case the rescaling function is the power law $\beta(\lambda) = \lambda^{1/2}$†.

In [3], Aurell *et al* use this system to model a velocity sequence, from which they compute the dissipation of energy.

More important to us is the example of a one-point turbulent signal, which is supposed to be self-similar with respect to the family of velocity differences, $\{\Delta_\tau : \Sigma \to \mathbb{R} : \tau \in \mathbb{N}\}$ such that $\Delta_\tau(\boldsymbol{v}) = \sum_{t=0}^{\tau-1} \boldsymbol{v}(t)$. In this case the space of sequences $\Sigma \subset \mathbb{R}^{\mathbb{N}}$ includes all the possible velocity sequences

$$\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t_0), \boldsymbol{v}(\boldsymbol{x}, t_0 + \mathrm{d}t), \boldsymbol{v}(\boldsymbol{x}, t_0 + 2\,\mathrm{d}t), \ldots$$

recorded in a fully developed turbulence experiment. An ergodic measure μ is assumed to exist, representing the statistical temporal behaviour of the system. Assuming self-similarity in the so-called inertial range, it is possible to deduce the self-similarity exponent from the statistical theory of turbulence‡, which gives $\alpha = \frac{1}{3}$.

How do we decide whether a given experimental sequence is the finite-time realization of a self-similar system? This question assumes the self-similarity of the sequence we analyse, when self-similarity has been deduced form a previous analysis or from a theoretical prediction as in the case of the one point turbulence. We need of course some information about the self-similarity range T and, the latter being essential, about the family of observables to consider. So we place ourselves in the situation where both the family of observables with respect to which our system is supposed to be self-similar and a subset of the self-similarity range, are known.

† This example belongs to a more general class where the Gaussian distribution is replaced by any zero-mean stable measure of exponent $\alpha \neq 1$, giving a scaling function $\beta(\lambda) = \lambda^{1/\alpha}$. We refer the reader to Feller [2] for this generalization.

‡ We refer to Landau and Lifshitz [4], for a deduction of this exponent, first due to Kolmogorov.

2. Empirical investigation of self-similarity

Let $\boldsymbol{v}$ be a sequence in Σ and $N \in \mathbb{N}$ the time length during which we record the sequence. We suppose that this sequence is a realization of a system which is self-similar with respect to the family $\{f_\tau : \Sigma \to \mathbb{R} : \tau \in \mathbb{N}\}$, in a finite range T_0. We will characterize each observable f_τ at finite time N by 'the empirical (N, τ)-maximum'

$$\delta_N(\tau) \equiv \max\{|f_\tau \circ \sigma^t(\boldsymbol{v})|,\ 0 \leqslant t \leqslant N\}. \tag{2.1}$$

For any finite subset T_0 of the self-similarity range, and thanks to the ergodic property stated in equation (1.1), the empirical (N, τ)-maximum approximately follows, for a large class of systems, the same scaling as the measure.

Proposition 2. Under some technical assumptions about the asymptotic distribution of f_τ values and the convergence rate of the empirical measures to the asymptotic distribution, for all τ, τ' in a finite subset of the self-similarity range T we have

$$\frac{\delta_N(\tau')}{\delta_N(\tau)} \to \left(\frac{\tau'}{\tau}\right)^\alpha \qquad \text{when } N \to \infty. \tag{2.2}$$

We postpone the proof, as well as the exact statement of this result, to the appendix.

Equation (2.2) allows us to compute a finite-time approximation for the self-similarity exponent in the following way.

Procedure 1. For the experimental sequence $\boldsymbol{v} \in \Sigma$ and the observation time $N \in \mathbb{N}$,

(i) compute $\delta_N(\tau)$ for all τ in the finite range T_0,

(ii) approximate the data $\log(\tau) \mapsto \log(\delta_N(\tau))$ for $\tau \in T_0$, by an affine function $\log(\tau) \mapsto \alpha_N \log(\tau) + b_N$, by linear regression. By doing so we obtain

$$\alpha_N = \frac{\sum_{\tau<\tau'\in T_0} \log(\tau'/\tau)\log(\delta_N(\tau')/\delta_N(\tau))}{\sum_{\tau<\tau'\in T_0} (\log(\tau'/\tau))^2}. \tag{2.3}$$

In this way we get an approximation α_N for the self-similarity exponent α.

Example 2.1. Consider a pseudo-random sequence of size $N = 5\times 10^4$, where each member is chosen independently, according to the Gaussian distribution of mean zero and variance one. We compute the maximum sum

$$\delta_N(\tau) = \max\left\{\left|\sum_{t=n}^{n+\tau-1} \boldsymbol{v}_t\right| : 0 \leqslant n \leqslant N\right\}$$

in the range $T = \{1, 2, \ldots, 20\}$, and compare it with the power function $\tau \mapsto \delta_N(1)\times\tau^{1/2}$ in figure 1. Applying the procedure described above, we find an approximation $\alpha_{10^4} = 0.4882$ for the self-similarity exponent $\alpha = \frac{1}{2}$.

Example 2.2. We analyse a turbulent wind tunnel sequence† with $R_\lambda = 180$, which is supposed to be a generic initial condition for a system self-similar on $T \subset \{20, 21, \ldots, 120\}$ with self-similarity exponent $\alpha = \frac{1}{3}$. The observation time is $N = 15 \times 10^4$ units of time‡. We compute the maximum difference

$$\delta_N(\tau) = \max\{|\boldsymbol{v}_{n+\tau} - \boldsymbol{v}_n| : 0 \leqslant n \leqslant N\}$$

† This turbulence sequence was obtained by the team of Anselmet to study the joint distribution of velocity and temperature [5].

‡ For an acquisition frequency of 37.5 kHz.

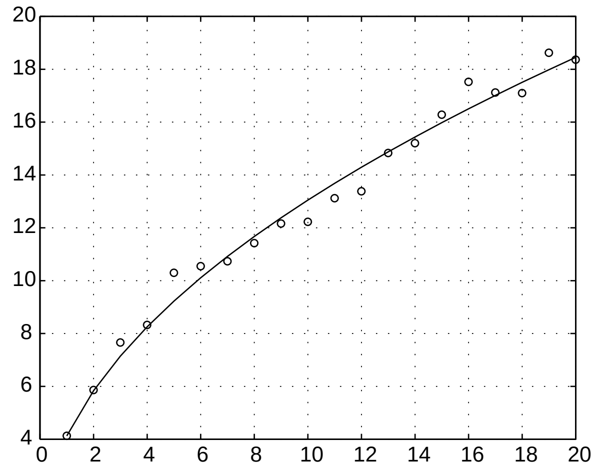

Figure 1. The circles represent the function $\tau \mapsto \delta_N(\tau)$ on the range $\tau \in \{1, 2, \ldots, 20\}$, computed from the random sequence of example 2.1 ($N \approx 10^4$). The full curve is the power law function $\tau \mapsto \delta_N(1) \times \tau^{1/2}$ for $\tau \in [1, 20]$.

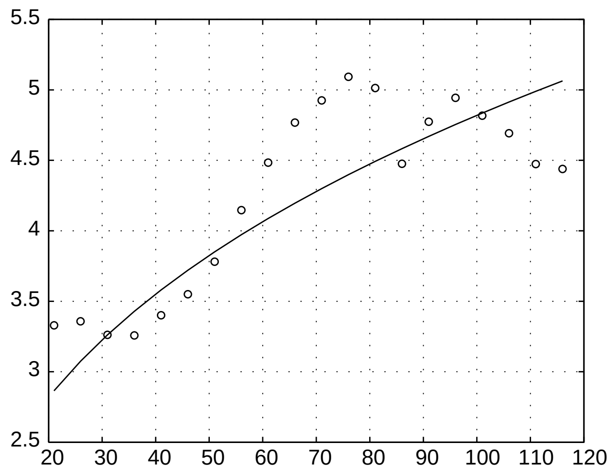

Figure 2. The circles represent the function $\tau \mapsto \delta_N(\tau)$ on the range $\tau \in \{21, 21+5, 21+10\ldots, 21+95\}$, computed from the turbulent sequence of Example 2.2 ($N \approx 10^5$). The full curve is the power law function $\tau \mapsto \delta_N(31) \times (\tau/31)^{1/3}$ for $\tau \in [21, 116]$.

in the range $T_0 = \{21, 21+5, 21+10, \ldots, 21+95\}$, and compare it with the power functions $\tau \mapsto \delta_N(21) \times (\tau/21)^{1/3}$ in figure 2. Applying the procedure to approximate the self-similarity exponent we find $\alpha_{15\times 10^4} = 0.2831$†.

Once we obtain an approximation for the self-similarity exponent we need a criterion to decide whether the analysed sequence may be considered as a realization of a self-similar system. What we propose is to compare the empiricalmeasures generated by the sequence,

† In the article of Anselmet *et al* [6], there is an interesting discussion about the convergence of the $(v_{n+\tau} - v_n)$-moments, for turbulent sequences.

using the exponent to renormalize these measures in the appropriate way.

For all τ in the self-similarity range T and $C > 0$ we define 'the C-normalized measure' μ_C such that for any measurable set $\mathcal{B} \subset \mathbb{R}$

$$\mu_C(\mathcal{B}) \equiv \mu\left\{\left|\frac{f_\tau \circ \sigma^t(\boldsymbol{v})}{C\ \tau^\alpha}\right| \in \mathcal{B}\right\} \tag{2.4}$$

which, because of the self-similarity of μ expressed in equation (1.2), does not depend on $\tau \in T$. In equation (2.4), the C-normalized measure, counts the asymptotic frequency for $|f_\tau/C\ \tau^\alpha|$ to take values in $\mathcal{B}$. Under the hypothesis ensuring the convergence, established in equation (1.1), the normalized C-measure of an interval $[k/P - 1/2P, k/P + 1/2P[$, can be uniformly approximated on T_0, by 'the empirical (P, τ, N)-normalized measure'

$\mu^{(P,\tau,N)} : \{0, 1, 2, \ldots, P\} \to [0, 1]$ such that

$$\mu^{(P,\tau,N)}(k) \equiv \frac{\mathrm{card}\{0 \leqslant t \leqslant N : \frac{2k-1}{2P} \leqslant |\frac{f_\tau \circ \sigma^t(\boldsymbol{v})}{\delta_N(\tau)}| < \frac{2k+1}{2P}\}}{N+1} \tag{2.5}$$

which gives the finite-time frequency for $|f_\tau/\delta_N(\tau)|$ to take values in one of the these intervals. The ergodic property for μ implies that $\mu^{(P,\tau,N)}$ must become independent of τ, when N goes to infinity. Since $\delta_N(\tau) \to C\tau^\alpha$ when $N \to \infty$ then, $\mu^{(P,\tau,N)}(k) \to \mu_C([k/P - 1/2P, k/P + 1/2P[)$ for some constant $C > 0$ independent of $\tau \in T_0$. In this way we obtain a criterion of self-similarity, which obviously does not necessarily imply self-similarity but nevertheless allows us to support or disregard, from the experimental point of view, the self-similarity hypothesis.

Self-similarity criterion. For an experimental sequence $\boldsymbol{v} \in \Sigma$ and a finite observation time $N \in \mathbb{N}$ we consider the difference

$$\|\mu^{(P,\tau,N)} - \mu^{(P,\tau',N)}\| \equiv \max\{|\mu^{(P,\tau,N)}(k) - \mu^{(P,\tau',N)}(k)| : 0 \leqslant k \leqslant P\} \tag{2.6}$$

between two empirical normalized measures and, given a candidate for the self-similarity exponent α, we define the deviation

$$h_N(\tau, \tau') \equiv |\log(\delta_N(\tau')/\delta_N(\tau)) - \alpha \log(\tau'/\tau)|. \tag{2.7}$$

For an experimental sequence that generates a self-similar system, for any τ in the self-similarity range, the difference $\|\mu^{(P,\tau,N)} - \mu^{(P,\tau',N)}\|$ between empirical measures, converges to an increasing function of the deviation $h_N(\tau, \tau')$, when $N \to \infty$. At finite N, it may happen that the correlation between $\|\mu^{(P,\tau,N)} - \mu^{(P,\tau',N)}\|$ and $h_N(\tau, \tau')$ does not define a function, or, if it does, that the resulting function oscillates. In any case the boundary functions

$$\eta \mapsto \max\{h_N(\tau, \tau') : \|\mu^{(P,\tau,N)} - \mu^{(P,\tau,N)}\| < \eta\} \tag{2.8}$$

$$\eta \mapsto \min\{h_N(\tau, \tau') : \|\mu^{(P,\tau,N)} - \mu^{(P,\tau,N)}\| > \eta\} \tag{2.9}$$

for $\tau, \tau' \in T_0$ and $\eta \in \mathbb{R}^+$, do not decrease. We consider that the corresponding experimental sequence is a realization of a self-similar system, if both boundary functions may be bounded, (2.8) from below and (2.9) from above, by a strictly increasing function. The increase of the boundaries stresses a real correlation between large deviations in the scaling with large differences between the empirical normalized measures. In this way the smaller the distance between the boundary functions is, the better the criterion is satisfied.

We apply this criterion to the sequences of examples (2.1) and (2.2). In the first case we take $\alpha = \frac{1}{2}$ and $T_0 = \{1, 2, \ldots, 20\}$. In the case of the turbulent sequence we test the

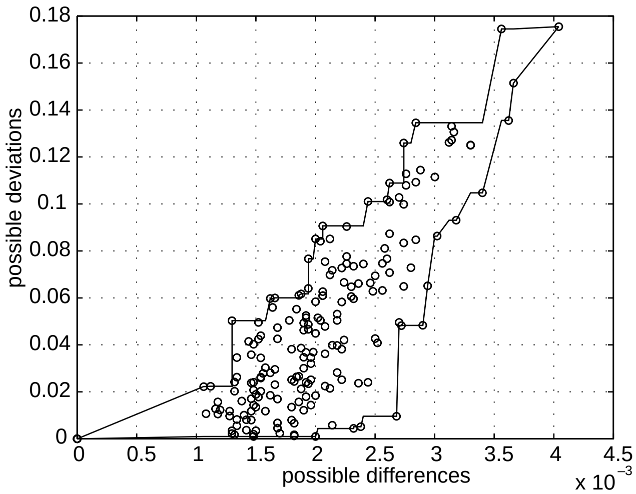

Figure 3. From the Gaussian sequence of the example 2.1 we plot the possible scaling deviations as a function of the possible differences on the empirical normalized measures,

$$\|\mu^{(P,\tau,N)} - \mu^{(P,\tau,N)}\| \to h_N(\tau,\tau') \equiv |\log(\delta_N(\tau')/\delta_N(\tau)(\tau/\tau')^{1/2})|$$

with $\tau, \tau' \in \{1, 2, \ldots, 20\}$ and $\|\mu^{(P,\tau,N)} - \mu^{(P,\tau,N)}\|$ as defined in equation (2.6). One circle at the point (η, h) represents a couple τ, τ' for which the difference between the empirical normalized measures is η and the deviation on the scaling is h. The full lines are the upper and lower boundary functions defined in equations (2.8) and (2.9), respectively.

criterion for $\alpha = \frac{1}{3}$ and $T_0 = \{21, 21+5, 21+10, \ldots, 21+95\}$. We show the results in figures 3 and 4 where we plot the relation†

$$\|\mu^{(P,\tau,N)} - \mu^{(P,\tau',N)}\| \mapsto h_N(\tau,\tau') \qquad \text{for } \tau, \tau' \in T_0$$

and its boundaries. We see that this criterion is satisfied by both sequences and that the relative difference between the boundaries is of the same order in both cases. We remark that this criterion does not depend on the partition P we use to define the empirical normalized measures and that the technical conditions ensuring the convergence in equation (2.2) are needed for the validity of this criterion. These conditions are satisfied for a large class of systems and it is difficult to imagine that the turbulence sequence behaves differently. On the other hand, the sequence of Gaussian random variables satisfies these conditions. Of course in real life the accuracy of the results depends on the way we generate the numerical random sequence.

Using this criterion of self-similarity we arrive at the following conclusion. The turbulent sequence of example 2.2 may be considered as the finite-time realization of a self-similar system with self-similarity exponent $\alpha \approx \frac{1}{3}$, and the sequence of Gaussian random variables of example 2.1 is a finite-time realization of the underlying theoretical model, with $\alpha = \frac{1}{2}$.

It is important to realize that our conclusions are relative to some theoretical model we control, because in general there is no *a priori* information either about the speed of convergence of the empirical measures or about the nature of the asymptotic one. It seems that for turbulent sequences with a larger Reynolds number, the time needed to observe convergence from the criteria we stated above is much larger than 15×10^4 time units, for an acquisition frequency like the one used for the sequence in example 2.2. In the

† Which is a function when τ or τ' is fixed.

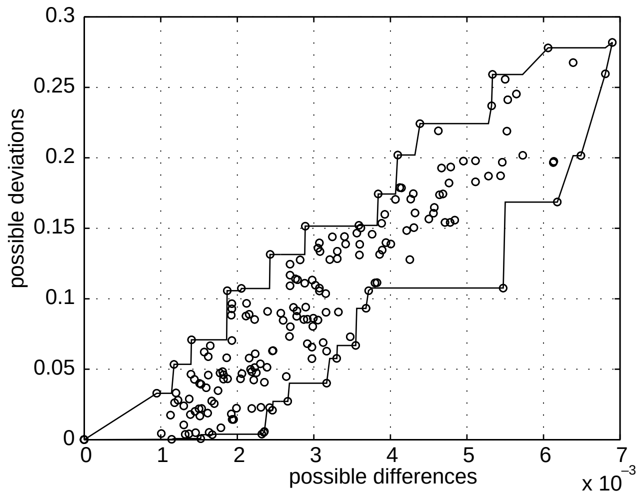

Figure 4. From the turbulent sequence of example 2.2 we plot the possible scaling deviations as a function of the possible differences on the empirical normalized measures,

$$\|\mu^{(P,\tau,N)} - \mu^{(P,\tau,N)}\| \to h_N(\tau,\tau') \equiv |\log(\delta_N(\tau')/\delta_N(\tau)(\tau/\tau')^{1/3})|$$

with $\tau, \tau' \in \{21, 21+5, 21+10, \ldots, 21+95\}$ and $\|\mu^{(P,\tau,N)} - \mu^{(P,\tau,N)}\|$ as defined in equation (2.6). One circle at the point (η, h) represents a couple τ, τ' for which the difference between the empirical normalized measures is η and the deviation on the scaling is h. The full lines are the boundary functions defined in equations (2.8) and (2.9), respectively.

theory of turbulence, since the work of Kolmogorov known as the K41 theory†, another way to characterize the self-similarity of such a system is well known and widely used. It is based on the fact that the scaling of the measure is reflected in the moments of the observables. Van Atta and Park [8] deduced, from the self-similarity, the power law behaviour of the moments. They studied the self-similarity of the experimental measures from direct comparison of the probability density functions, concluding that a quasi self-similarity exists in a restricted range for τ and $v_{n+\tau} - v_n$. In the next section we describe how the finite-time deviation may be the source of anomalous scaling, when the conditions of convergence for the empirical measures are satisfied. A discussion about the deviations caused by temporal intermittency closes the section.

3. Anomalous scaling from finite time effects

If the measurement time is large enough, the self-similarity property must already be present in the observations; however, deviations due to finite time effects may modify the expected behaviour. We can estimate these finite time effects in the case of a self-similar system for which the normalized measure μ_C satisfies a monotonicity property which we state below.

When a deviation on the power law behaviour of $\delta_N(\tau)$ is found, for instance if $\delta_N(\tau') < \delta_N(\tau)(\tau'/\tau)^\alpha$ when $\tau < \tau'$ then, the distance between the empirical probability measures $\mu^{(P,\tau,N)}$ and $\mu^{(P,\tau',N)}$ grows according to the self-similarity criterion. This difference implies that the 'empirical structure functions' defined below deviate from their asymptotic behaviour in the form of an 'anomalous scaling' in the sense of Paladin and

† All books about turbulence dedicate some pages to the K41 theory. Frisch [7] offers a view where the self-similarity hypotheses made in the original work are reformulated in a modern version.

Vulpiani [9].

The structure functions $\tau \mapsto S_q(\tau)$ account for the behaviour of the f_τ asymptotic moments, for f_τ in a family of observables $\{f_\tau : \Sigma \mapsto \mathbb{R} : \tau \in \mathbb{N}\}$. They follow a power law behaviour, for a self-similar system, on the self-similarity range T. This follows immediately from the similarity property stated in equation (1.2). In fact, 'the qth structure function' is the function $S_q : \mathbb{N} \to \mathbb{R}^+$ such that

$$S_q(\tau) \equiv \lim_{N\mapsto\infty} \frac{\sum_{t=0}^{N} |f_\tau(\boldsymbol{v}^t)|^q}{N+1} = \int_\Sigma |f_\tau(\boldsymbol{v})|^q \,\mathrm{d}\mu(\boldsymbol{v}) \tag{3.1}$$

and because of the similarity property, for all $\tau, \tau' \in T$ $(S_q(\tau')/S_q(\tau)) = (\tau'/\tau)^{\alpha\, q}$. The same was deduced by Van Atta and Park [8], from a hypothesis stronger than our self-similarity property but equivalent to it in the case of measures which are absolutely continuous with respect to the Lebesgue measure. In terms of the normalized measure μ_C introduced in equation (2.3), we can write the structure functions as

$$S_q(\tau) = (C\tau^\alpha)^q \times \int_{\mathbb{R}^+} x^q \,\mathrm{d}\mu_C(x). \tag{3.2}$$

In the self-similarity range, the integral in the last equation is independent of τ and then, for any $\tau, \tau' \in T$, the self-similarity exponent α satisfies

$$\log(S_q(\tau)) = (\alpha q)\log(\tau) + \log\left(C^q \times \int_{\mathbb{R}^+} x^q \,\mathrm{d}\mu_C(x)\right). \tag{3.3}$$

If the observation time is large enough, one would be tempted to use this property to determine the self-similarity exponent of the system as the slope of the affine function $\log(\tau) \mapsto \log(S_q(\tau))$ defined on T. To this end we need to define the finite-time–finite-precision version of the structure functions.

For a finite observation time $N \in \mathbb{N}$ and a finite precision $P \in \mathbb{N}$ we define 'the empirical structure function' $S_{(q,N)} : \mathbb{N} \to \mathbb{R}^+$, such that

$$S_{(q,N)}(\tau) \equiv \left(\frac{\delta(\tau)}{P}\right)^q \sum_{k=0}^{P} \mu^{(P,\tau,N)}(k) k^q \tag{3.4}$$

computed from the empirical (P, τ, N)-normalized measure defined in equation (2.4), which is the finite-time–finite-precision version of a C-normalized measure μ_C for $\delta_N(\tau) \approx C\tau^\alpha$.

For N large enough, the 'empirical structure function' follows an approximate power law inside T, allowing us to approximate $q\ \alpha$ by the slope of the best affine approximation to the empirical function $\log(\tau) \mapsto \log(S_{(q,N)}(\tau))$, for τ in a finite range $T_0 \subset T$. This defines, in the same way as Anselmet *et al* determine the scaling exponents in [6], a function of $q \mapsto \chi_N(q)$ which we call 'the empirical scaling law', such that

$$\chi_N(q) = \frac{\sum_{\tau<\tau'\in T_0} \log(S_{(q,N)}(\tau')/S_{(q,N)}(\tau)) \log(\delta_N(\tau')/\delta_N(\tau))}{\sum_{\tau<\tau'\in T_0} (\log(\tau'/\tau))^2}. \tag{3.5}$$

We think of this function as the finite-time–range-dependent approximation to the asymptotic scaling law $q \mapsto \alpha\ q$, which is always linear for a self-similar system. Equation (3.5) is obtained by linear regression of the data $\log(\tau) \mapsto \log(S_{(q,N)}(\tau)))$, for $\tau \in T_0$. We may already decompose this quantity into a linear and a nonlinear component,

$$\chi_N(q) = \alpha_N q + f_N(q) \tag{3.6}$$

where $\alpha_N(q)$ is the finite-time approximation to the scaling exponent we defined in equation (2.3), and

$$f_N(q) = \frac{\sum_{\tau<\tau'\in T_0} \log(\tau'/\tau) D_{\tau'\to\tau}(q)}{\sum_{\tau<\tau'\in T_0} (\log(\tau'/\tau))^2} \tag{3.7}$$

with

$$D_{\tau'\to\tau}(q) \equiv \log\left(\frac{\sum_{k=0}^{P}\mu^{(P,\tau',N)}(k)(k/P)^q}{\sum_{k=0}^{P}\mu^{(P,\tau,N)}(k)(k/P)^q}\right) \tag{3.8}$$

the nonlinear component of $\chi_N(q)$, which we call the 'correction function'. This is the key quantity for the study of anomalous scaling from finite-time deviations. The interest for this function comes from the statistical theory of turbulence, and we hope that the conclusions we obtain can be extended to other systems where the same kind of phenomenon appears. For some time the self-similarity property was supposed to hold in turbulence, but further experimental research and refined theoretical assumptions led to a new interpretation of the experimental results. Nowadays it is generally accepted that the structure functions of the velocity differences $\{\Delta_\tau(\boldsymbol{v}) : \tau \in \mathbb{N}\}$ follow an anomalous scaling law in some range of values of τ, instead of the self-similarity property for the Δ_τ values implying 'normal scaling' (in opposition to anomalous) for the structure functions. Let us specify what we understand by anomalous scaling.

A family of observables $\{f_\tau : \Sigma \to \mathbb{R} : \tau \in \mathbb{N}\}$ possesses anomalous scaling with respect to the measure μ, in the range $T \subset \mathbb{N}$, if for all $q \in \mathbb{N}$ there exists a constant C_q and a nonlinear function $q \mapsto \chi(q)$ such that

$$\int_\Sigma |f_\tau(\boldsymbol{v})|^q \, \mathrm{d}\mu(\boldsymbol{v}) = C_q \times \tau^{\chi(q)}. \tag{3.9}$$

In fact $q \mapsto \chi(q)$ turns out to be a concave function thanks to the Hölder inequality. From the experimental point of view, the anomalous exponent $\chi(q)$ is computed in the same way as we compute $\chi_N(q)$, and because of the decomposition in equation (3.6) one may think of the nonlinearity of the function $q \mapsto \chi(q)$ as having a finite-time origin, a longer observation time being needed to put into evidence the self-similar nature of the phenomena. The convergence time will then be determined by the convergence of α_N (the empirical self-similarity exponent), depending on the collective behaviour of the empirical maxima, rather than by the convergence of the moments†. In the last part of this section we prove that, under certain conditions on the asymptotic normalized measure, the correction function introduced in equation (3.7) is positive for all q, and increasing and concave for small values of q. In this way the mechanism leading to anomalous scaling will be perceptible only for finite-time measurements, whereas the underlying ergodic measure may still be considered as a self-similar measure. By taking this point of view, we can establish a relation between finite-time anomalies and the temporal intermittency of the velocity sequence‡. We will come back to this relation in the next section.

The knowledge of the relative deviation between two empirical measures allows us to determine some properties of the correction function introduced in equation (3.7), when the asymptotic normalized measure satisfies the following monotonicity property.

In the self-similarity range T, the normalized measures are related in such a way that $\mu_C(\mathcal{A}) = \mu_{C'}(C/C'\mathcal{A})$, for any measurable set $\mathcal{A} \subset \mathbb{R}^+$ and $C, C' > 0$. We say that μ_C is 'simply decreasing' if for all precisions $P \in \mathbb{N}$ and for all deviations $\lambda > 0$ in the normalization constant C, there exists a single intersection point $p(P,\lambda) \in \mathbb{N}$ and a minimal

† The convergence of moments was the criterion used by Anselmet *et al* [6].

‡ We understand temporal intermittency in the sense of Pomeau and Manneville [10].

difference $\eta(P, \lambda) > 0$ between the corresponding normalized measures, that is

$$\mu_C\left(\left[\frac{2k-1}{2P}, \frac{2k+1}{2P}\right[\right) < \mu_C\left(\left(1+\frac{\lambda}{C}\right)\left[\frac{2k-1}{2P}, \frac{2k+1}{2P}\right[\right) + \eta \quad \text{when } 0 \leqslant k < p$$

$$\mu_C\left(\left[\frac{2k-1}{2P}, \frac{2k+1}{2P}\right[\right) + \eta > \mu_C\left(\left(1+\frac{\lambda}{C}\right)\left[\frac{2k-1}{2P}, \frac{2k+1}{2P}\right[\right) \quad \text{when } p < k \leqslant P. \tag{3.10}$$

Notice that the crossing point p can be larger than P. In that case the second inequality never takes place. These inequalities mean that the normalized measures μ_C and $\mu_{C/(1+\lambda)}$, when they are described up to a finite-precision P, cross at a single point and, a long way from this intersection, the distance between them is bounded from below. This is the case for a Gaussian or exponentially decreasing measure, and for some other monotonously decreasing measures. It seems that the asymptotic measure corresponding to the turbulent data of example 2.2 also satisfies this property, since its empirical normalized measure satisfies an analogous property, described in proposition 3. In fact, if the observation time is large enough, the property of simple decreasing is inherited by the empirical normalized measure.

Proposition 3. Let the normalized measure μ_C, for a self-similar system with self-similarity exponent α, be simply decreasing. Let $\tau, \tau' \in T_0$, with T_0 a finite subset of the self-similarity range.

For any precision $P \in \mathbb{N}$ and a deviation $\lambda > 0$ there exists an observation time $N_0 \in \mathbb{N}$ such that, if the deviation $\delta_N(\tau')/\tau'^{\alpha} < \delta_N(\tau)/\tau^{\alpha} + \lambda$ is found, then

$$\mu^{(P,N,\tau)}(k) < \mu^{(P,N,\tau')}(k) \qquad \text{when } 0 \leqslant k < p$$

$$\mu^{(P,N,\tau)}(k) > \mu^{(P,N,\tau')}(k) \qquad \text{when } p < k \leqslant P \tag{3.11}$$

for some $p \leqslant P$ and all $N \geqslant N_0$.

Proof. By the ergodicity of the system, for all $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that for all $N \geqslant N(\epsilon)$

$$\left|\mu_{C(\tau)}\left(\left[\frac{2k-1}{2P}, \frac{2k+1}{2P}\right[\right) - \mu^{(P,N,\tau)}(k)\right| < \epsilon$$

with $C(\tau) = \delta_N(\tau)/\tau^{\alpha}$. On the other hand, the simple decreasing property of μ_C implies the existence, for any precision P, of a crossing point p and a minimal distance η between $\mu_{C(\tau')}$ and $\mu_{C(\tau)}$, which depends on $\lambda = C(\tau) - C(\tau')$. Taking $\epsilon < \eta/2$ we obtain (3.11), for all $N \geqslant N_0$ and some $p \in \mathbb{N}$. We only need to ensure that $p < P$. This is the case because

$$\sum_{k=0}^{P} \mu^{(P,N,\tau)}(k) = \sum_{k=0}^{P} \mu^{(P,N,\tau')}(k) = 1$$

implying that $\mu^{(P,N,\tau)}(k)$ cannot be larger than $\mu_{(P,N,\tau')}(k)$ for all $0 \leqslant k \leqslant P$. □

Now we describe the behaviour of the correction function $f_N(q)$ introduced in equation (3.7), in terms of the deviations in the empirical normalized measures. These are due to deviations from the asymptotic scaling behaviour on the empirical maxima δ_N.

From equation (3.7) it is clear that the behaviour of the correction function is controlled by the difference $D_{\tau' \to \tau}(q)$ defined in (3.8). In the range of validity of proposition 3, this difference follows a well-defined behaviour depending on the deviations of the corresponding empirical maxima, obtained thanks to the properties of simply decreasing measures. Indeed, to each difference $D_{\tau'\to\tau}(q)$ corresponds a couple of maxima $\delta_N(\tau'), \delta_N(\tau)$, which determine its behaviour.

Proposition 4. Let μ_C be a simply decreasing measure such that for all $\lambda > 0$ there exists a minimal distance $\eta > 0$ and an associated precision P_0 such that

$$\mu_C\left(\left[\frac{1}{2P}, \frac{3}{2P}\right[\right) - \mu_C\left(\left(1+\frac{\lambda}{C}\right)\left[\frac{1}{2P}, \frac{3}{2P}\right[\right) > \eta \qquad \forall\ P \geqslant P_0 \tag{3.12}$$

where μ_C is the normalized measure for a self-similar system with self-similarity exponent α.

Then, there exists a minimal precision $P_{\min} \geqslant 0$, such that for all precisions $P \geqslant P_{\min}$ and all deviations $\lambda > 0$ there exists an observation time N_0 such that, for all $N \geqslant N_0$, if $\delta_N(\tau')/\tau'^\alpha < \delta_N(\tau)/\tau^\alpha + \lambda$, then

(i) $D_{\tau'\to\tau}(q) \geqslant 0$ for all $q \geqslant 0$.

(ii) There exists a minimal precision $P_0 > 0$ such that, if the precision P used to compute $q \mapsto D_{\tau'\to\tau}(q) \geqslant 0$ is larger than P_0, then there exists $q_0 > 0$ such that $D_{\tau'\to\tau}(q)$ is a concave function on the interval $[0, q_0]$.

Proof. The difference $D_{\tau'\to\tau}(q)$ corresponding to an observation time N and a precision $P \in \mathbb{N}$ can be written as

$$D_{\tau'\to\tau}(q) = \log\left(1 + \frac{\sum_{k=0}^{P}(\mu^{(P,N,\tau')}(k) - \mu^{(P,N,\tau)}(k))(k/P)^q}{\sum_{k=0}^{P}\mu^{(P,N,\tau)}(k)(k/P)^q}\right)$$

and $q \mapsto D_{\tau'\to\tau}(q)$ is positive if and only if

$$\sum_{k=0}^{P}(\mu^{(P,N,\tau')}(k) - \mu^{(P,N,\tau)}(k))\left(\frac{k}{P}\right)^q \geqslant 0. \tag{3.13}$$

Recall that we use the same hypothesis as in proposition 3, so for all deviations λ and precisions P there exists an observation time N_0 such that for all $N \geqslant N_0$ inequality (3.10) holds. In this case

$$\sum_{k=1}^{P}\mu^{(P,N,\tau')}(k) - \mu^{(P,N,\tau)}(k) = 0 - (\mu^{(P,N,\tau')}(0) - \mu^{(P,N,\tau)}(0)) > 0.$$

That shows the validity of (3.13) for $q = 0$. Let us denote by $\mu^{(P,N,\tau'\to\tau)}(k)$ the difference $\mu^{(P,N,\tau')}(k) - \mu^{(P,N,\tau)}(k)$ for all $k \in \mathbb{N}$, and suppose that at the intersection point $0 < p < P$, one has the inequality $\mu^{(P,N,\tau')}(p) < \mu^{(P,N,\tau)}(p)$†. We decompose the sum on the left-hand side of (3.13) into a positive and a negative part‡,

$$\begin{array}{ccc}
\displaystyle\sum_{k=0}^{p}\mu^{(P,N,\tau'\to\tau)}(k)\left(\frac{k}{P}\right)^q & + & \displaystyle\sum_{k=p+1}^{P}\mu^{(P,N,\tau'\to\tau)}(k)\left(\frac{k}{P}\right)^q \\
\downarrow\geqslant & & \downarrow\geqslant \\
\displaystyle\left(\frac{p}{P}\right)^q\left(\sum_{k=0}^{p}\mu^{(P,N,\tau'\to\tau)}(k)\right) & + & \displaystyle\left(\frac{p}{P}\right)^q\left(\sum_{k=p+1}^{P}\mu^{(P,N,\tau'\to\tau)}(k)\right) > 0.
\end{array}$$

† The other two possibilities, $\mu^{(P,N,\tau')}(p) = \mu^{(P,N,\tau)}(p)$ or $\mu^{(P,N,\tau')}(p) > \mu^{(P,N,\tau)}(p)$, lead to the same result.
‡ The symbol $\downarrow\geqslant$ means that the upper term is greater than the lower one in a vertical array. We use it to stress the term by term inequality.

In this way we prove part (i) of this proposition. To prove the second part, that concerns the concave increasing, we find after some easy calculations that in order for $(\mathrm{d}/\mathrm{d}q)\ D_{\tau'\to\tau}(q)$ to be positive it is sufficient that

$$\mathrm{d}/\mathrm{d}q\left(\sum_{k=0}^{P}\mu^{(P,N,\tau'\to\tau)}(k)\left(\frac{k}{P}\right)^q\right) > 0$$

and for $(\mathrm{d}/\mathrm{d}q)^2 D_{\tau'\to\tau}(q)$ to be negative it is sufficient that

$$(\mathrm{d}/\mathrm{d}q)^q\left(\sum_{k=0}^{P}\mu^{(P,N,\tau'\to\tau)}(k)\left(\frac{k}{P}\right)^q\right) < 0.$$

For each one of these derivatives we decompose the resulting sum into a positive and a negative part, and then evaluate the result at $q = 0$. We obtain

$$\lim_{q\to 0^+}\frac{\mathrm{d}}{\mathrm{d}q}\left(\sum_{k=0}^{P}\mu^{(P,N,\tau'\to\tau)}(k)\left(\frac{k}{P}\right)^q\right)$$
$$= \begin{array}{ccc} \displaystyle\sum_{k=1}^{p}\mu^{(P,N,\tau'\to\tau)}(k)\log\left(\frac{k}{p}\right) & + & \displaystyle\sum_{k=p+1}^{P-1}\mu^{(P,N,\tau'\to\tau)}(k)\log\left(\frac{k}{p}\right) \\ \downarrow\geqslant & & \downarrow\geqslant \\ \log(P)\mu^{(P,N,\tau'\to\tau)}(1) & + & \log\left(\dfrac{P-1}{P}\right) \end{array}$$

and

$$\lim_{q\to 0^+}\frac{\mathrm{d}^2}{\mathrm{d}q^2}\left(\sum_{k=0}^{P}\mu^{(P,N,\tau'\to\tau)}(k)\left(\frac{k}{P}\right)^q\right)$$
$$= \begin{array}{ccc} \displaystyle\sum_{k=1}^{p}\mu^{(P,N,\tau'\to\tau)}(k)\log\left(\frac{k}{p}\right)^2 & + & \displaystyle\sum_{k=p+1}^{P-1}\mu^{(P,N,\tau'\to\tau)}(k)\log\left(\frac{k}{p}\right)^2 \\ \downarrow\leqslant & & \downarrow\leqslant \\ \log(P)^2\mu^{(P,N,\tau'\to)}(1) & + & \log\left(\dfrac{P-1}{P}\right)^2. \end{array}$$

At this point we need to make use of the hypothesis stated in inequality (3.12), which holds for a normalized measure continuous with respect to Lebesgue. It is sufficient to take a precision P greater than P_0 and

$$P^* \equiv \max(\exp(\log(2)^{1/2}/(\eta-2\epsilon)^{1/2}), \exp(\log(2)/(\eta-2\epsilon)))$$

which depends on the minimal difference η, and on the distance $\epsilon < \eta/2$ between empirical and asymptotic normalized measures. Finally, for all deviations $\lambda > 0$ and $P \geqslant P_{\min} \equiv \max(P_0, P*)$, there exists an observation time N_0 such that, if $\delta_N(\tau')/\tau'^{\alpha} < \delta_N(\tau)/\tau^{\alpha} + \lambda$, then $(\mathrm{d}/\mathrm{d}q)D_{\tau'\to\tau}(0) > 0$ and $(\mathrm{d}/\mathrm{d}q)^2 D_{\tau'\to\tau}(0) < 0$. Thanks to the continuity of the derivatives of $D_{\tau'\to\tau}(q)$, there exists an interval $[0, q_0]$, such that $(\mathrm{d}/\mathrm{d}q)D_{\tau'\to\tau}(q) > 0$ and $(\mathrm{d}/\mathrm{d}q)^2 D_{\tau'\to\tau}(q) < 0$ for all $q \in [0, q_0]$ around zero. □

This proposition allows us to describe the behaviour of the correction function through the differences $D_{\tau'\mapsto\tau}(q)$.

We can find an interval of values $[0, q_0]$ where, for each couple $\tau < \tau' \in T_0$ such that $\delta_{N(\epsilon)}(\tau') < (\tau'/\tau)^{\alpha} \times \delta_{N(\epsilon)}(\tau)$, there is a positive, increasing concave contribution to the correction function, whereas this contribution is negative, decreasing convex if $\delta_{N(\epsilon)}(\tau') > (\tau'/\tau)^{\alpha} \times \delta_{N(\epsilon)}(\tau)$. The equilibrium between positive and negative deviations determines the nature of the correction function and it seems impossible to have an exact cancellation between positive and negative contributions. That explains the nonlinearity of the empirical scaling law at finite time in the general case.

Let us remark that the final shape of the correction function generally depends on the observation time through the empirical maxima, and for a fixed observation time, on the choice of the self-similarity range T_0. The simple decreasing property defined by inequality (3.11), and the hypothesis stated in (3.12) are essential for the validity of proposition 4. They are satisfied by the Gaussian measure and several other measures defined by decreasing densities. With respect to the turbulent sequences, the behaviour of the empirical normalized measures supports the hypothesis of an asymptotic normalized measure satisfying these properties. In the following section we restrict the study to measures with these characteristics.

4. Temporal intermittency and finite-time deviations

We saw that finite-time deviations from self-similarity may produce an appearance of anomalous scaling, because of finite-time nonlinear corrections to the asymptotic scaling law. From the analysis of turbulence sequences, once the similarity range is determined via the third structure function†, the determination of the empirical scaling law shows, in all the reported cases, a well defined deviation. Usually the empirical scaling law $\chi_N(q)$ is compared with concave, asymptotically affine increasing functions, coming from models of the distribution of energy dissipation in space as in She and Leveque [11] or Novikov [12], or from symmetry considerations as in Dubrulle and Graner [13]. It was Kolmogorov [14] who first proposed a nonlinear concave function to model the nonlinearity of $\chi_N(q)$, from a model for the distribution of energy dissipation. Nevertheless this is an inconsistent model as noted by Mandelbrot [15]. In [16] we relate this kind of behaviour of $\chi_N(q)$ with the properties of the dynamical system generated by the corresponding sequence, without any assumption about asymptotic self-similarity.

In some cases, when this concave nonlinear scaling law is derived from an energy cascade model‡, the notion of intermittency is appealed to explain the deviation from a linear scaling law, but this notion remains quite ambiguous in that framework. From our point of view, the relation between this deviation and intermittency really exists and can be stated in an explicit way. In fact, if we assume that the turbulent sequence is the superposition of a sequence generating a self-similar system on the Kolmogorov's inertial range and of another sequence, which is temporally intermittent in the sense of Pomeau and Manneville [10], we can show that the resulting empirical scaling law is composed of a linear part with slope $\alpha_N < \frac{1}{3}$ and of a correction function $f_N(q)$ which is always positive and increasingly concave on an interval $[0, q_0]$.

To simplify our description let us consider that the intermittent sequence

$$\boldsymbol{I} = \boldsymbol{I}_1, \boldsymbol{I}_2, \ldots, \boldsymbol{I}_t, \ldots \in [-V, V]^{\mathbb{N}}$$

with long sequences of zeros alternating with a single large oscillation $u_1, u_2, \ldots, u_k$ such that $u_n \in [-V, V]$ for $1 \leqslant n \leqslant k$, is superimposed on the self-similar sequence

$$\boldsymbol{v} = \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_t, \ldots \in \mathbb{R}^{\mathbb{N}}$$

† Since, from the K41 theory, $S_3(\tau) \propto \tau$ on the inertial range, our self-similarity range.

‡ See Paladin and Vulpiani [9] and references therein.

which generates a system, self-similar on T with respect to the family $\{\Delta_\tau : \mathbb{R}^{\mathbb{N}} \to \mathbb{R} : \tau \in \mathbb{N}\}$, with similarity exponent $\alpha > 0$. In these circumstances, the observed sequence is

$$\boldsymbol{v} \oplus \boldsymbol{I} \in \mathbb{R}^{\mathbb{N}} \qquad \text{such that } \boldsymbol{v} \oplus \boldsymbol{I}_t = \begin{cases} \boldsymbol{v}_t & \text{if } \boldsymbol{I}_t = 0 \\ \boldsymbol{I}_t & \text{if } \boldsymbol{I}_t \neq 0. \end{cases} \tag{4.1}$$

When the intermittent sequence begins to oscillate, it replaces the usual self-similar sequence. One can characterize the intermittent sequence $\boldsymbol{I}$ by the temporal frequency of the values $\boldsymbol{I}_t$. The state $\boldsymbol{I}_t = 0$ corresponds to the so-called 'laminar regime', while $\boldsymbol{I}_t, \boldsymbol{I}_{t+1}, \ldots, \boldsymbol{I}_{t+k-1} = u_1, u_2, \ldots, i_k$ represents the 'chaotic regime' on the intermittent models as in Wang [17] or Collet *et al* [18]. Let us simply remark that the empirical measures generated by $\boldsymbol{I}$ converge to the measure concentrated at zero, that is

$$\lim_{N\to\infty} \frac{\#\{0 \leqslant t \leqslant N : \sigma^t(\boldsymbol{I}) \in \mathcal{B}\}}{N+1} = \begin{cases} 0 & \text{if } \mathbf{0} \notin \mathcal{B} \\ 1 & \text{if } \mathbf{0} \in \mathcal{B} \end{cases}$$

for any measurable set $\mathcal{B} \subset \mathbb{R}^{\mathbb{N}}$, where $\mathbf{0} \in \mathbb{R}^{\mathbb{N}}$ is the sequence formed only by zeros, that is, $\mathbf{0}_t = 0$ for $t \in \mathbb{N}$.

Hypothesis about the superposition.

(i) We consider that the empirical measure generated by $\boldsymbol{v} \oplus \boldsymbol{I}$ converges to the same measure as $\boldsymbol{v}$, that is

$$\lim_{N\to\infty} \frac{\#\{0 \leqslant t \leqslant N : \sigma^t(\boldsymbol{V}) \in \mathcal{B}\}}{N+1} = \lim_{N\to\infty} \frac{\#\{0 \leqslant t \leqslant N : \sigma^t(\boldsymbol{v}) \in \mathcal{B}\}}{N+1} \tag{4.2}$$

for all measurable sets $\mathcal{B} \subset \mathbb{R}^N$. This limit satisfies the self-similarity property, whereas the empirical maxima associated with it can be greatly modified with respect to those associated with $\boldsymbol{v}$.

(ii) The maximal differences associated with $\boldsymbol{I}$ have a negative deviation from the asymptotic scaling, that is, there exists $\lambda > 0$ such that for all couples $\tau < \tau' \in T_0$,

$$\frac{\max\{|u_{n+\tau'} - u_n| : 1 \leqslant n \leqslant k - \tau'\}}{\max\{|u_{n+\tau} - u_n| : 1 \leqslant n \leqslant k - \tau\}} < \left(\frac{\tau'}{\tau}\right)^\alpha - \eta \qquad \text{for all } \tau, \tau' \in T_0. \tag{4.3}$$

Equation (4.2) ensures, for any precision P and for the deviation η appearing in the last inequality, the existence of an observation time N_0 such that for all $N > N_0$ propositions 3 and 4 hold for the empirical measure generated by $\boldsymbol{v} \oplus \boldsymbol{I}$.

Theorem 1. Let us suppose that the superposition $\boldsymbol{v} \oplus \boldsymbol{I}$ of a self-similar and an intermittent sequence satisfies the hypothesis stated above. There exists a minimal precision $P_{\min}$ and for all $P \geqslant P_{\min}$ there exists a minimal observation time N_0 such that, if for all observation times $N_0 \leqslant N \leqslant N_0 + K$, $K > 0$ we have

$$\delta_N(\tau) = \max\{|u_{n+\tau} - u_n| : 1 \leqslant n \leqslant k - \tau\} \qquad \forall\ \tau \in T_0 \tag{4.4}$$

where k is the length of the intermittent oscillation, and the empirical maximum $\delta_N(\tau)$ is computed from $\boldsymbol{v} \oplus \boldsymbol{I}$. Then the empirical scaling law $\chi_N(q)$, computed from $\boldsymbol{v} \oplus \boldsymbol{I}$ with precision P, decomposes as

$$\chi_N(q) = \alpha_N(q)q + f_N(q)$$

where $\alpha_N < \alpha$ and $q \mapsto f_N(q)$ is a positive function, concave and increasing in a range $[0, q_0]$, and that for all $N_0 \leqslant N \leqslant N_0 + K$, $K > 0$.

Proof. This result follows directly from the decomposition of $\chi_N(q)$ established in equation (3.6),

$$\chi_N(q) = \alpha_N \, q + f_N(q) = \frac{\sum_{\tau<\tau'\in T_0} \log(\tau'/\tau)\log(\delta_{N_0}(\tau')/\delta_N(\tau))}{\sum_{\tau<\tau'\in T_0}(\log(\tau'/\tau))^2} q + \frac{\sum_{\tau<\tau'\in T_0}\log(\tau'/\tau) D_{\tau'\to\tau}(q)}{\sum_{\tau<\tau'\in T_0}(\log(\tau'/\tau))^2}.$$

The hypothesis in (4.3) and (4.4) ensures that for any couple $\tau, \tau' \in T_0$ and $N_0 \leqslant N \leqslant N_0 + K$,

$$\log\left(\frac{\delta_N(\tau')}{\delta_N(\tau')}\right) = \log\left(\frac{\max\{|u_{n+\tau'} - u_n| : 1 \leqslant n \leqslant k - \tau'\}}{\max\{|u_{n+\tau} - u_n| : 1 \leqslant n \leqslant k - \tau\}}\right) < \log\left(\left(\frac{\tau'}{\tau}\right)^{\alpha} - \lambda\right)$$

where the maxima $\delta_N(\tau)$ are computed from the superposition $\boldsymbol{v} \oplus \boldsymbol{I}$. In this way we obtain $\alpha_N < \alpha$.

In proposition 4, which applies to the sequence $\boldsymbol{v} \oplus \boldsymbol{I}$ thanks to the hypothesis in (4.2), we stated the existence of a minimal precision $P_{\min}$ such that for all $P \geqslant P_0$ and all deviations $\lambda > 0$, there exists a minimal observation time N_0 such that for all $N \leqslant N_0$ the difference functions $q \mapsto D_{\tau'\to\tau}(q)$ computed at those times are positive for all $q \geqslant 0$, and concave increasing in a range $[0, q_0]$. Since $f_N(q)$ is a positive linear combination of such differences, the linear part of χ_N also has these properties. □

In this way we construct a 'model sequence' for which the asymptotic behaviour agrees with the self-similar predictions, but for which the finite-time deviations give to $\chi_N(q)$ 'anomalous' characteristics.

Given a precision $P > P_{\min}$ and for observation times $N_0 \leqslant N \leqslant N_0 + K$, the empirical scaling law decomposes as a sum

$$\chi_N(q) = \alpha_N \, q + f_N(q)$$

where the empirical scaling exponent α_N is smaller than the asymptotic one, and the nonlinear correction function $f_N(q)$ is always positive and concave increasing in a range $[0, q_0]$. In this range, the empirical scaling law is a nonlinear, concave increasing function, bounded from below by a linear increasing function $q \mapsto \alpha_N q$. This is exactly the same behaviour as predicted by the energy cascade models cited above, with the difference that in this case it is valid only for finite times and the concavity is ensured only on a finite range of q.

The next step of this investigation about the possible finite-time origin for the anomalous scaling is to confront this simplified model of superposition of self-similarity and intermittency, with the experiment. In practice, if the turbulent sequence can be decomposed as the superposition of a self-similar and an intermittent component, the characteristics of the intermittent one may not be as trivial as we have supposed. We took the oscillations on the intermittent sequence as being always of the same type, while in a real experiment they may not be. In this way the behaviour of $\delta_N(\tau)$ may sometimes be dominated by the intermittent oscillation and at some other times by the self-similar sequence itself, depending on the observation time and/or the values of τ we consider.

5. Conclusions

We have discussed two questions related to the finite-time behaviour of a sequence that generates a self-similar system. The first one concerns the way to approach the self-similar exponent associated with the system, from the analysis of the sequences observed during a finite time. After this approximation procedure we established a criterion to decide whether the analysed sequence may be considered to be a finite-time subsequence of a generating sequence for a self-similar system. This criterion regarding the properties of the empirical measure generated by the sequence at a fixed time, the observation time, does not imply that the sequence really generates a self-similar measure, simply because it is impossible to give sufficient conditions to ensure it, when we regard only finite-time realizations. Nevertheless it is essential to relate the finite-time realization to some asymptotic known behaviour, otherwise it is impossible to develop our analysis consistently. The main motivation of this work is the investigation of self-similarity on the system generated by a turbulent sequence, implied by the postulates of Kolmovorov's theory of turbulence and by the Taylor hypothesis [19]. This prediction was already considered in the work of Van Atta and Park [8], unfortunately their analysis is mostly focused on the search for non-self-similar effects, and some interesting observations deserving a further investigation were not developed. Using the criterion of self-similarity on both a Gaussian sequence and a turbulent sequence, we arrive at the conclusion that the same degree of self-similarity must be assigned to both sequences, although only in the first case can we be sure that the system is self-similar. Then the observations of Van Atta and Park about the lack of self-similarity mainly for extreme events, leading to anomalous scaling in the structure functions, may be thought of as a finite-time effect. That leads to the formulation of the second question. We suppose that a given sequence is really a finite-time realization of a self-similar system, and we look for the conditions ensuring anomalous scaling at finite time, taking into account the particular kind of deviation on the scaling law found on the analysis of turbulent data. In fact, the empirical scaling laws reported on the literature, despite the precise value for the exponents, all have in common a concave increasing shape, which is related to 'intermittency' on the energy cascade models (Novikov [12], Frisch [20], She and Leveque [11] and others). To introduce this particular kind of deviations, given a finite-time realization which is compatible with the self-similar asymptotic behaviour, we suppose that an intermittent effect, with no incidence on the measure convergence, but modifying the behaviour of the empirical maxima, is superimposed on the self-similar sequence. In this way the scaling law obtained from such a sequence possesses the same type of deviation as the one predicted by some of the energy cascade models.

It remains to develop an analysis capable of putting into evidence this kind of temporal intermittent effect and allowing a decomposition of the turbulent signal into a real self-similar sequence superimposed on an intermittent one. It is also interesting to apply this kind of reasoning to other systems where anomalous scaling may be related to finite-time effects.

Acknowledgments

This work was partially supported by the *Université de la Méditerranée* by an ATER contract. We thank all the members of the Dynamical Systems team at CPT-Marseille for their interesting comments and suggestions at each level of the realization of this work, the anonymous referees for their interesting remarks on the manuscript and Fabien Anselmet for giving us access to the experimental turbulent sequence we used in example 2.2.

Appendix

One of the consequences of the self-similarity property, introduced in equation (1.2) is the fact that under certain circumstances, the scaling functions $\tau'/\tau \mapsto \beta(\tau'/\tau)$ follow a power law.

Proposition 5. Let (Σ, σ, μ) be a self-similar system with respect to $\{f_\tau : \Sigma \mapsto \mathbb{R} : \tau \in \mathbb{N}\}$, with self-similarity range T and scaling function $\tau'/\tau \mapsto \beta(\tau'/\tau)$. If for all $\tau \in T$, $\mu \circ f_\tau^{(-1)}$ is not the atomic measure at 0, then $\beta(\lambda) = \lambda^\alpha$ for some $\alpha \in \mathbb{R}$.

Proof. Let τ, τ', and $\tau'' \in T$. By similarity we have

$$\mu\{f_{\tau''} \in \mathcal{B}\} = \mu\{f_\tau \in \beta(\tau/\tau'')\mathcal{B}\}$$
$$\mu\{f_{\tau''} \in \mathcal{B}\} = \mu\{f_{\tau'} \in \beta(\tau'/\tau'')\mathcal{B}\}$$
$$\mu\{f_{\tau'} \in \beta(\tau'/\tau'') \times \mathcal{B}\} = \mu\{f_\tau \in \beta(\tau/\tau')\beta(\tau'/\tau'')\mathcal{B}\}$$

and then,

$$\mu\{f_\tau \in \beta(\tau'/\tau'')\beta(\tau/\tau')\mathcal{B}\} = \mu\{f_\tau \in \beta(\tau/\tau'')\mathcal{B}\}$$

for any Borel set $\mathcal{B} \subset \mathbb{R}$, τ, τ' and $\tau'' \in T$.

Let us take $\mathcal{B} = B_\epsilon(0)/\beta(\tau/\tau'')$ with $B_\epsilon(0) = \{x \in \mathbb{R} : |x| \leqslant \epsilon\}$, then $\mu\{\Delta_\tau \in B_\epsilon(0)\} = \mu\{\Delta_\tau \in \ell B_\epsilon(0)\}$, with

$$\ell = \frac{\beta(\tau'/\tau'')\beta(\tau/\tau')}{\beta(\tau/\tau'')}.$$

If $\ell \neq 1$,

$$\mu\{f_\tau \in B_\epsilon(0)\} = \mu\{f_\tau \in B_{\epsilon\ell^n}(0)\} \qquad \forall\, \epsilon > 0$$

for all $\epsilon > 0$, $n \in \mathbb{N}$, which implies that $\mu \circ f_\tau^{-1}$ is the measure concentrated at 0. This being impossible by hypothesis,

$$\beta(\tau/\tau'') = \beta(\tau'/\tau'')\beta(\tau/\tau') \qquad \forall\, \tau, \tau', \tau'' \in T.$$

Then, $\beta(\lambda) = \lambda^\alpha$ for some $\alpha \in \mathbb{R}$. □

Proposition 6. Let (Σ, σ, μ) be a self-similar system with respect to $\{f_\tau : \Sigma \to \mathbb{R} : \tau \in \mathbb{N}\}$, with self-similarity range T and exponent α.

Thanks to the ergodic property stated in equation (1.1), for all $\epsilon > 0$ there exists an observation time $N(\epsilon) \in \mathbb{N}$ such that

$$\left|\frac{\#\{0 \leqslant t \leqslant N : |f_\tau \circ \sigma^t(v)| \in \mathcal{B}\}}{N+1} - \mu\{|f_\tau| \in \mathcal{B}\}\right| < \epsilon \tag{H.1}$$

for all $\tau \in T_0$, $N \geqslant N(\epsilon)$ and any measurable set $\mathcal{B} \subset \Sigma$, T_0 being a finite subset of the self-similarity range T. Let us suppose that this convergence is such that

$$\lim_{\epsilon \to 0} \epsilon N(\epsilon) = 0. \tag{H.2}$$

Under these hypotheses, for all τ, τ' in T_0,

$$\frac{\delta_N(\tau')}{\delta_N(\tau)} \to \left(\frac{\tau'}{\tau}\right)^\alpha \qquad \text{when } N \to \infty.$$

Proof. For $\epsilon > 0$ and all $\tau \in \mathbb{N}$ we define

$$\delta_\epsilon(\tau) \equiv \inf\{R \in \mathbb{R}^+ : \mu\{|f_\tau(v)| \geqslant R\} > \epsilon\}.$$

Thanks to the self-similarity property,

$$\mu\{|f_\tau(v)| \geqslant R\} = \mu\{|f_\tau(v)| \geqslant (\tau'/\tau)^\alpha \times R\} \qquad \forall\, \tau, \tau' \in T$$

which implies that

$$\frac{\delta_\epsilon(\tau')}{\delta_\epsilon(\tau)} = \left(\frac{\tau'}{\tau}\right)^q. \tag{T.1}$$

The ergodic property stated in (H.1) implies that

$$\text{if } \#\{0 \leqslant t \leqslant N(\epsilon) : |f_\tau \circ \sigma^t(\boldsymbol{v})| > R\} = 0 \text{ then } \mu\{|f_\tau| \geqslant R\} < \epsilon$$

and from here we deduce that $\delta_\epsilon(\tau) \leqslant \delta_{N(\epsilon)}(\tau)$, for all $\tau \in T_0$.

On the other hand, if $\mu\{|f_\tau| \geqslant R\} < \epsilon$ for some $R > 0$, then

$$\#\{0 \leqslant t \leqslant N(\epsilon) : |f_\tau \circ \sigma^t(\boldsymbol{v})| \in \mathcal{B}\} < 2\epsilon(N(\epsilon) + 1).$$

Hypothesis (H.2) implies the existence of $\epsilon_c > 0$ such that for all $\epsilon \leqslant \epsilon_c$, $2\epsilon(N(\epsilon)+1) < 1$, which implies $\#\{0 \leqslant t \leqslant N(\epsilon) : |f_\tau \circ \sigma^t(\boldsymbol{v})| \in \mathcal{B}\} = 0$.

In this way we finally get $\delta_{N(\epsilon)}(\tau) = \delta_\epsilon$, for all $\epsilon < \epsilon_c$ or equivalently, for all $N \geqslant N(\epsilon_c)$. Replacing δ_ϵ by $\delta_{N(\epsilon)}$ in (T.1) the proposition is proved. □

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